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Is there a better coordinate system out there?

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Curious Calculus: Is there a better coordinate system out there?

Author 1 (James—pseudonym for student)
Author 2 (Ken—pseudonym for teacher)

Part I: Ken as Teacher

Introduction

Reflecting on 25 years of classroom teaching, I have fond memories of helping students build valuable problem-solving skills. Some of my most treasured experiences come from a surprising place—the honest questions that student ask. Teachers volley all kinds of questions; some questions can be resolved quickly while others consume far more time and energy. The questions that are truly unforgettable are those that initially terrified me as a teacher—these are the questions for which *I did not have an answer*. Early in my career, questions like this made me feel inadequate—did I have a gap somewhere in my knowledge? Would students interpret my hesitation as incompetence? With time, these suspicions proved incorrect. As the questions continued to pour in, I conveyed to students that this form of open inquiry is a natural part of learning. If students were asking questions I couldn't immediately answer, this signaled to me that I was doing my job! Students were thinking deeply about the content. They were curious about generalizing. They were wondering out loud. This is precisely what it means to nurture a culture of learning and apprenticeship. Recently, I revisited Dan Teague's words from years past:

> How will [students] learn to do things [teachers] can't? How will they become better than us? After all, if our students aren't better than their teachers, then we are moving backward. One of my prime directives in teaching is to not let the limitations of my talents be limitations on theirs." (Teague, 2015).

Teague reminds us that good teaching is really a balance of skill, motivation, and humility.

These early experiences began an interesting journey for me, a teaching colleague (###### #####) and many, many of my students. I began collecting these questions—I called them "Curious Calculus Questions"—and answering them in class, in office hours, at department meetings, at conferences, or one-on-one with the student. Questions came at all levels. Some were intuitive and easy to answer, once we pondered them briefly. For example,

1. (Calculus I)[1] $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ or $\frac{d}{dr}\left(A_{\text{circle}}\right) = C_{\text{circle}}$. Is this a neat coincidence or something deeper?

2. (Calculus II) Can an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ still converge without the condition $a_{n+1} \le a_n$?

Other questions were more difficult to answer; we needed to consult branches of mathematics I had long forgotten. These questions often had surprising answers:

[1] I've indicated in parentheses when/where the question is likely to be asked.

3. (Calculus II) If $\int_1^\infty f(x)\,dx$ converges, must $\lim_{x\to\infty} f(x)=0$?

4. (Calculus II) Similar to $\frac{0}{0}$ and $\frac{\infty}{\infty}$, is $\frac{\text{oscillation}}{\text{oscillation}}$ an indeterminate form?

Of course, none of the above investigations are too challenging with enough stubbornness and determination. However, the reader should remember that these questions came from (beginning) students enrolled in Calculus I and Calculus II.

For the interested reader, answers to the above questions 1-4 (and many others) can be found in Author (2021, 2024) and Author et al. (2016, 2017a, 2017b). After investigating these questions and finding solutions, we would sometimes indulge in a literature search to see if anyone else had pondered something similar. The journals *Mathematics Teacher* (National Council of Teachers of Mathematics) and the *College Mathematics Journal* (Mathematical Association of America) often publish articles with content related to our questions. Sometimes we found answers to the same questions we tackled (item 1—Perrin & Quinn, 2008; Zazkis, Sinitsky, & Leikin, 2013). On occasion, we discovered more advanced mathematical treatments of our question (item 4—Fink & Sadek, 2013). Even if someone halfway across the world had written about a similar situation a decade prior, it rarely diminished the student's joy of the journey. Quite the opposite—it validated the experience. On more than one occasion, students had shared with me that investigating their specific curious calculus question was more exciting than anything they had done in their four years of college.

A Thorny Question is Born

One question that remains open in my class came from a Calculus III student more than 20 years ago. Shortly after learning about cylindrical and spherical coordinates, this student asked me directly, "Are there other coordinate systems out there? You know, ones that might work for just one or two things?" My instinct was "yes" but like many questions I was asked, I simply did not know. That semester, I offered his question as an extra credit assignment: *Can you invent a new coordinate system and convince us of its worth?* As the years passed (and I continued to teach Calculus III), I offered this assignment as an opportunity for students to try something new and creative. Eventually, my request morphed into an optional classroom assignment that very few students attempted. Students were pessimistically direct when I asked why they chose not to complete it. For example,

1. *This is different. It is too hard*.
2. *Actually, I attempted it. I thought for hours but I have nothing to show for it*.

Students quickly realized this wasn't just another random extra credit assignment where the tutoring center could save the day. Instead, I was asking students to break new ground. Everyone was genuinely stymied.

As time passed, even if students didn't attempt the problem, they still contributed to everyone's collective understanding of the problem. Some joked, "I like that you can call the system whatever you want—hyperbolic coordinates." This comment was made in reference to an attempt to incorporate a hyperbolic paraboloid into the coordinate system. A different student enjoyed the freedom in notation—"I can use (r,a,t) [my initials] for the letters." Others wondered aloud, "Can I use three angles sort of like rectangular [coordinates] uses 3 lengths?" Each year, I felt gradual and steady movement toward a viable coordinate system. I recall one student saying, "This is totally a legit difficult problem. You could have a ridiculous integral, change coordinates with the Jacobian we learned, and then have a simple integral." This student was sharing with his classmates that not only would it be fun to invent a coordinate system, but it was a *functionally meaningful* exercise to do so. This student's realization was eventually incorporated into the assignment I still give today (see Figure 1).

THE TASK: Broadly speaking, you are asked to create your own three-dimensional coordinate system and to explain some of its advantages and/or disadvantages.

In an attempt to motivate this, think of the three systems that you already know:

(a) (x, y, z): the rectangular coordinate system

(b) (r, θ, z): the cylindrical coordinate system

(c) (ρ, θ, ϕ): the spherical coordinate system

Each has its merits depending on the setting in which it is used. Likewise, each has its own set of ills!

Some thoughts to get you going:

(a) Define clearly the letter(s) you are using. Give precise definitions of each. Also provide a picture so everything makes sense. Is there a set of equations to move from this system to rectangular? And vice versa?
(b) Why is this a good system? Where/When might it not be so good? Try to compare/contrast with the three systems above, if appropriate.
(c) Think about integration in this new system. What is the Jacobian? Are certain integrals easy to evaluate in this new system but challenging elsewhere? Give an example or two.

Figure 1. Text of the assignment I give to Calculus III students.

Despite years of failed attempts, this problem had such widespread appeal that the Mathematics Club (which I co-advise) spent several meetings pondering the problem to see if they could find a solution. We held some wonderful discussions, shared many laughs, but came up mostly empty. As a club, we even built a wooden 3-space model from $(0,0,0)$ to an arbitrary (x, y, z) to facilitate the process. We used painter's/duct tape with different colors to indicate different ways of traveling from the origin to (x, y, z). For example, in Figure 2, the reader can see that the black tape is the standard Cartesian coordinate system while the blue tape was a student's attempt to answer the question, "Do we even *need* three degrees of freedom in 3D? Could we do it with just two lengths?"

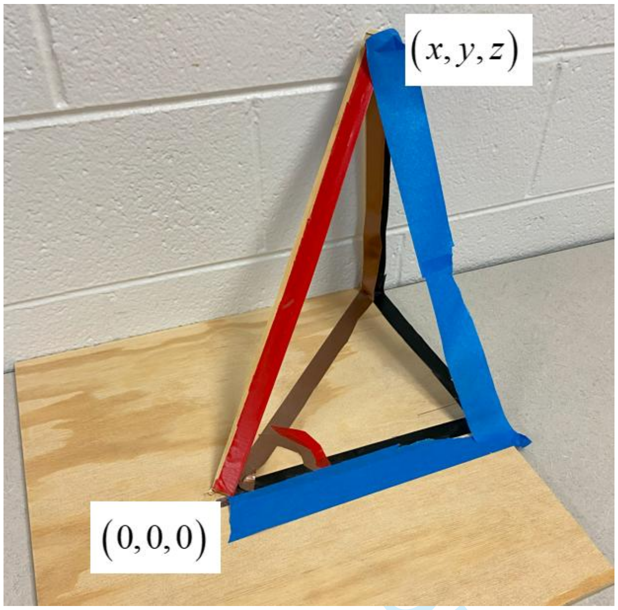

Figure 2. A wooden model built by and discussed in Math Club to motivate thinking about a new coordinate system.

For the remainder of this article, we share the fruits of one student's (James) efforts from 2024-2025. This section details the conversations we held (teacher and student), as well as the student's motivation, missteps, revisions, and eventual resolution. After discussing the coordinate system, both James and Ken share some of the benefits from having taken this journey together.

Part II: James as Student

Diamond Coordinates (James's Attempt at a New Coordinate System)

When presented with the (optional) task of creating a new coordinate system, I was instantly intrigued. My professor introduced the problem and shared that many of his teaching colleagues were convinced this problem was too difficult for Calculus III students. Right then and there, I wanted to solve this problem and prove his colleagues wrong.

The first step was lots of brainstorming and thinking. Eventually the core idea of my system boiled down to this: Rather than polar coordinates lying on some circle, what if they laid on some other shape? I played around with different shapes, and I finally settled on a diamond. Figure 2 shows some of my initial analysis while Figure 3 shows how I had conceived of this new system as related to polar coordinates. At this point, my professor mentioned its similarity to the taxicab/city grid metric for measuring distance in metropolitan areas (see, "Taxicab geometry," 2025). Despite my work not being "new" mathematics, we both agreed it would be interesting to try to determine the equations to move from Cartesian to Diamond and vice versa, and to follow this with determining the Jacobian transformation. I was on my way!

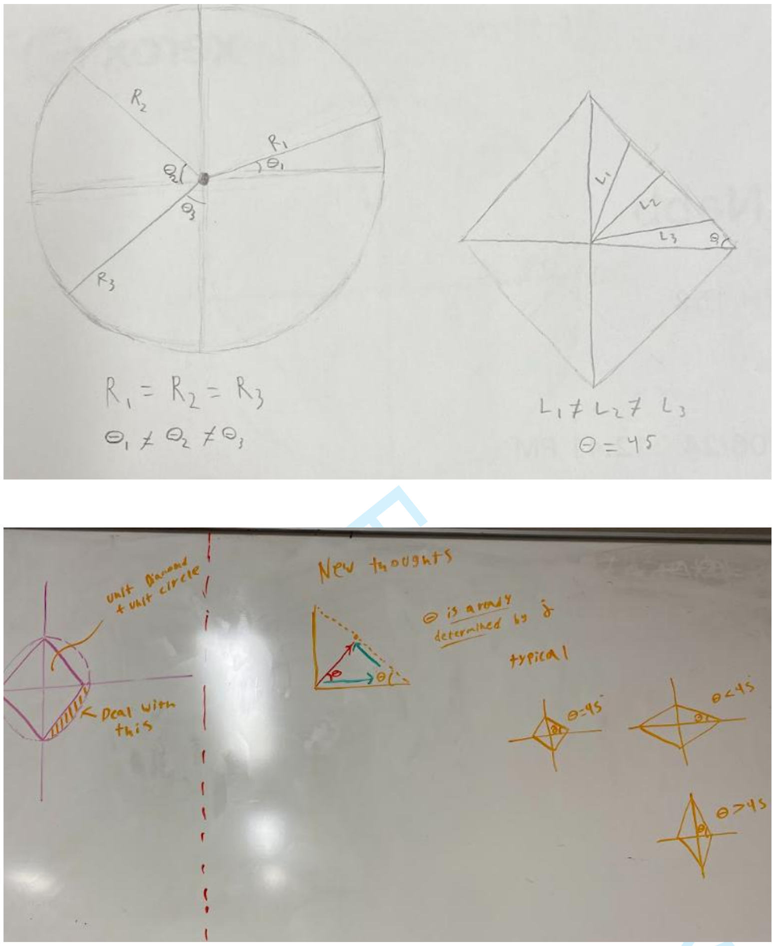

Figure 2. A sample of James's early thinking and analysis, along with some early whiteboard discussions with Ken.

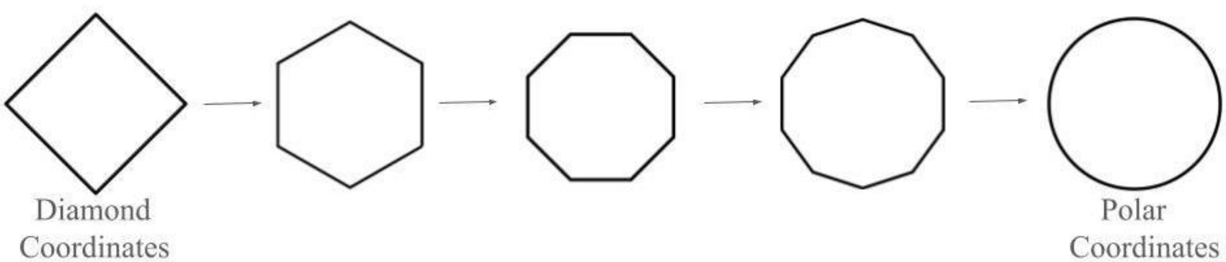

Figure 3. Connecting Diamond Coordinates to the familiar Polar Coordinates.

Here is how I conceived of this new system. See Figure 4. Keeping point A arbitrary, I simply shifted the angle and the length. Instead of r and θ, I now had p and 45°. This resulted in a new variable length p and a fixed angle of 45°, meaning point A fell on a unique line segment (or diamond, depending on the quadrant of A). Initially, I labeled the distance from the origin to a corner of the diamond and called it *j*, pronounced *ya*[2]. However, like polar coordinates, I wanted *j* (or *j*) to act like a radial length, which meant that $x + y = j$ in Quadrant I. With time, my new coordinate system (*j, p*) was born, and it had these definitions:

> j = the directed distance from the origin to an arbitrary point on the diamond, and
> p = the measure of how far along (as a percentage) point *A* is located along the specific diamond which passes through it, with the assumed starting position on the positive *x*-axis.

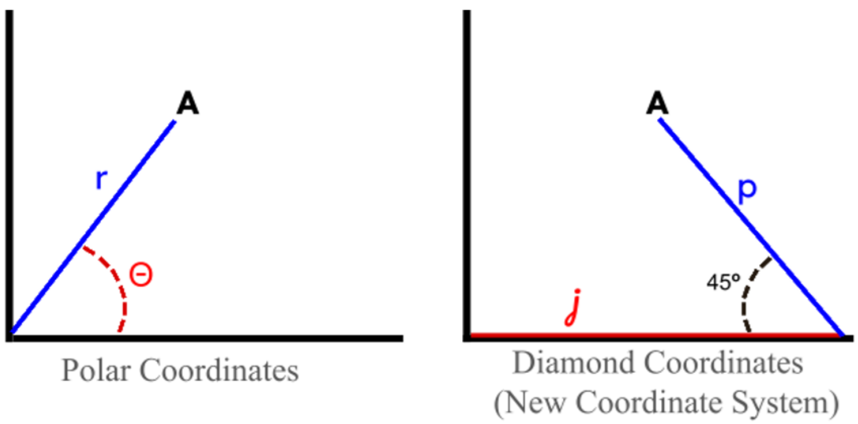

Figure 4. My early thinking on using a length and an angle.

One realization that came about is how challenging it is to construct clear definitions of what something truly means! Given the way *j* is defined, it is clear we can determine the mathematical relationship between *j*, *x*, and *y*. In Quadrant I, we have $x + y = j$; in Quadrant II, we get $-x + y = j$, and so on. Generalizing, we obtain $|x| + |y| = j$. My initial thinking on a unit diamond can be seen in Figure 5, followed by the more developed and refined thinking in Figure 6.

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[2] We eventually decided to change *j* to the letter *j* since we encountered typesetting restrictions in mathematical environments.

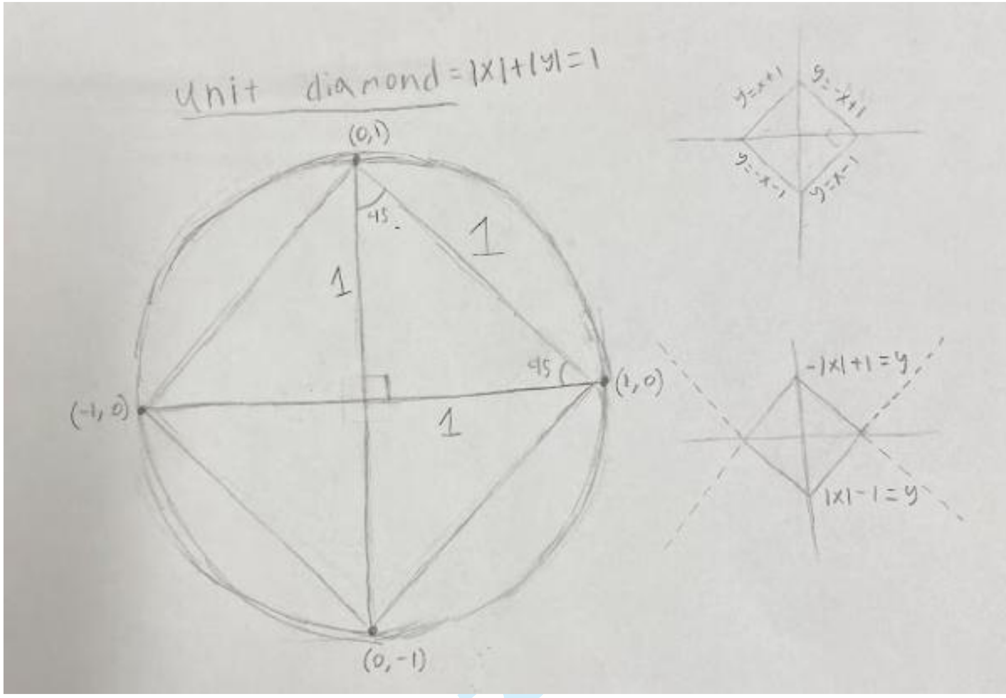
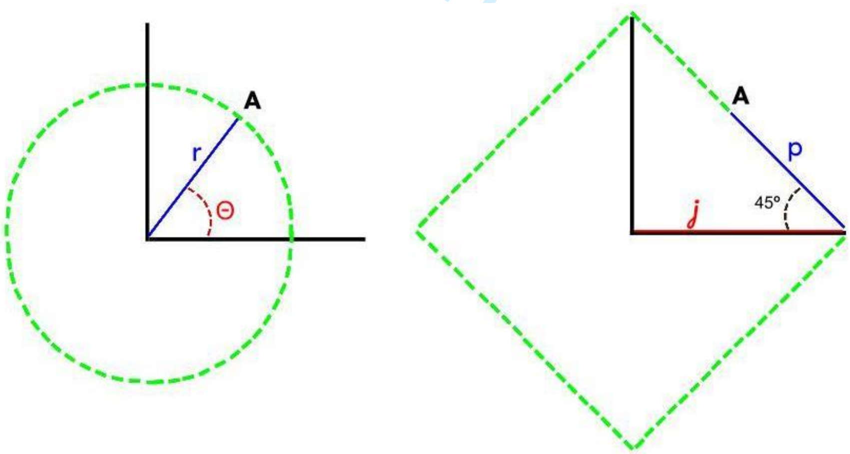


Figure 5. Finding the relationship between *j*, *x*, and *y* for a specific case.



$x^2 + y^2 = r^2$ $\qquad$ $|x| + |y| = j$

Figure 6. The similarities and differences between Polar Coordinates (left) and what I eventually called Diamond Coordinates (right).

The quantity p represents how far along point A is located along the diamond, assuming a starting position on the positive x-axis. Phrased differently, p can be interpreted as a percentage around the diamond, so ranging in values from 0 to 1, where 0 represents 0% and 1 represents 100%. For example, the p shown in Figure 6 has $p \approx 12.5\% = 0.125$. I discussed with Ken how the p value is analogous to running around the bases in the sport of baseball, except that baseball's first base represents the starting point ($p = 0$) and ending point ($p = 100$) for the coordinate system, much like the standard $\theta = 0$ and $\theta = 2\pi$ indicate equivalent angular positions in Polar Coordinates. With this convention, we see that $\theta = 2\pi p$ or $p = \frac{\theta}{2\pi}$. So with a little bit of work, I was able to derive two relationships

$$|x| + |y| = j \tag{1}$$

$$p = \frac{\theta}{2\pi} \tag{2}$$

where I could use what I know about θ from polar coordinates—namely, $\tan \frac{y}{x} = \theta$, $\cos\theta = \frac{x}{r}$, and $\sin\theta = \frac{y}{r}$, where $r = \sqrt{x^2 + y^2}$. The connection between what I created to what I already knew was incredibly exciting!

In order to visualize the *x*-*y* connection to the *j*-*p* connection, I wrote a Desmos script to understand what is being measured and calculated: https://www.desmos.com/calculator/co0og89saq. The sliders in the program allow the user to change the size of the radial diamond (*j*) and increase/decrease how far point *A* progresses (*p*) along the diamond. The complicated equations in the animation are briefly discussed below.

Once I had a working coordinate system that defined every point in the plane, I wanted to calculate the Jacobian matrix $J = \begin{vmatrix} \partial x / \partial j & \partial x / \partial p \\ \partial y / \partial j & \partial y / \partial p \end{vmatrix}$ for the purposes of integral transformation. Would it be something simple like r in polar coordinates or $\rho^2 \sin\phi$ in spherical coordinates? I first had to determine $x = x(j, p)$ and $y = y(j, p)$. This was a very tall order—involving lots of (acrobatic) algebraic manipulations. Using $|x| + |y| = j$ and $p = \frac{\theta}{2\pi}$, I obtained

$$x = \operatorname{sgn}(\cos(2\pi p)) \frac{j|\cos(2\pi p)|}{|\cos(2\pi p)| + |\sin(2\pi p)|} \tag{3}$$

and

$$y = \operatorname{sgn}(\sin(2\pi p)) \frac{j|\sin(2\pi p)|}{|\cos(2\pi p)| + |\sin(2\pi p)|}, \tag{4}$$

where sgn is the signum function $\operatorname{sgn}(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$ (Sign function, 2025). I had to look that up! For the reader interested in the algebraic details, see Appendix 1 at the end of this article. Once this was complete, I obtained the four partial derivatives below:

$$\frac{\partial x}{\partial j}=\operatorname{sgn}(\cos(2\pi p))\frac{|\cos(2\pi p)|}{|\cos(2\pi p)|+|\sin(2\pi p)|} \tag{5}$$

$$\frac{\partial y}{\partial j}=\operatorname{sgn}(\sin(2\pi p))\frac{|\sin(2\pi p)|}{|\cos(2\pi p)|+|\sin(2\pi p)|} \tag{6}$$

$$\frac{\partial x}{\partial p}=\operatorname{sgn}(\cos(2\pi p))\times \frac{(|\cos(2\pi p)|+|\sin(2\pi p)|)\frac{j\cos(2\pi p)}{|\cos(2\pi p)|}-j|\cos(2\pi p)|\left(\frac{\cos(2\pi p)}{|\cos(2\pi p)|}+\frac{\sin(2\pi p)}{|\sin(2\pi p)|}\right)}{(|\cos(2\pi p)|+|\sin(2\pi p)|)^2} \tag{7}$$

$$\frac{\partial y}{\partial p}=\operatorname{sgn}(\sin(2\pi p))\times \frac{(|\cos(2\pi p)|+|\sin(2\pi p)|)\frac{j\sin(2\pi p)}{|\sin(2\pi p)|}-j|\sin(2\pi p)|\left(\frac{\cos(2\pi p)}{|\cos(2\pi p)|}+\frac{\sin(2\pi p)}{|\sin(2\pi p)|}\right)}{(|\cos(2\pi p)|+|\sin(2\pi p)|)^2} \tag{8}$$

For details on $\frac{\partial x}{\partial p}$, see Appendix 2. In all its glory, the Jacobian is

$$\begin{aligned}J=&\operatorname{sgn}(B)\frac{(|A|+|B|)\frac{jB}{|B|}-j|B|\left(\frac{A}{|A|}+\frac{B}{|B|}\right)}{(|A|+|B|)^2}\left(\operatorname{sgn}(A)\frac{|A|}{|A|+|B|}\right)\\ &-\operatorname{sgn}(A)\frac{(|A|+|B|)\frac{jA}{|A|}-j|A|\left(\frac{A}{|A|}+\frac{B}{|B|}\right)}{(|A|+|B|)^2}\left(\operatorname{sgn}(B)\frac{|B|}{|A|+|B|}\right)\end{aligned} \tag{9}$$

where $A=\cos(2\pi p)$ and $B=\sin(2\pi p)$. At this point, I suspected that the integral transformation would likely serve little to no purpose given the complexity of the Jacobian. Thus, my focus has shifted to whether I can simplify the Jacobian to a manageable form; maybe this will disclose some relationship that is masked by its algebraic complexity? After several months of work, this is where this journey has taken me. This problem and its potential application continue to be on my mind to this day.

Student Reflections (James)

This assignment was meaningful to me because it lit a mathematical fire that will not extinguish anytime soon. I am still actively working on this problem. At the time of my initial work, I did not prepare for final exams and I did not complete my weekly homework. While I was working on this coordinate system, it was the most motivated I had ever felt as a student.

Educators are like tour guides through an ever expansive forest of knowledge. Allowing students to venture off the beaten path and explore new ideas is the goal. When making this coordinate system, my professor didn't know the answers, nor did I. We held some great conversations and it felt like I was carving out a unique little island of math—something I could call my own. This was uncharted territory where I was defining my own variables, choosing appropriate notation, and creating my own system. This problem transformed me from a passive learner into an active thinker. I can only hope that my college journey continues to provide similar experiences as I explore more advanced coursework and determine the focus of my studies. Long live mathematics!

Teacher Reflections (Ken)

A career is a long time, even if you love what you do. I have been in mathematics classrooms for 25 years. Some moments in one's career are singular—yes, like a comet on a hyperbolic orbit. I realize that this is probably one of those moments. For many years, I offered this problem as a challenging enrichment activity alongside the standard sea of "exercises" we see in the mathematics curriculum. James is the only student to have ever meaningfully explored this challenge.

A few things have left a lasting impression on me. First, seeing James's excitement throughout the process has been fun to watch. James refused to give up. I know this is not Fermat's Last Theorem but to James, it sure felt like it. I can only hope that months (or maybe years) from now, I receive a surprise email message from James with the subject line, "I got it!" This is what makes mathematics so unpredictably fun, yet teachers rarely get to share this level of mathematical joy with their students. Second, we had many discussions on how to explain the meaning of these quantities in a way where an "outsider"—someone just glancing at this coordinate system for the first time—could understand what James was thinking. Constructing unambiguous definitions is a *fantastically* difficult ordeal (Van Dormolen & Zaslavsky, 2003; Vinner 1991) and James was able to experience this. Finally, I still occasionally ask myself: Did James "solve" the problem? This is for the reader to decide. But to all the teachers out there, I wouldn't give him a grade just yet. There is still time. If I have learned anything in this process, it is that we likely haven't heard the last of the Diamond Coordinate system.

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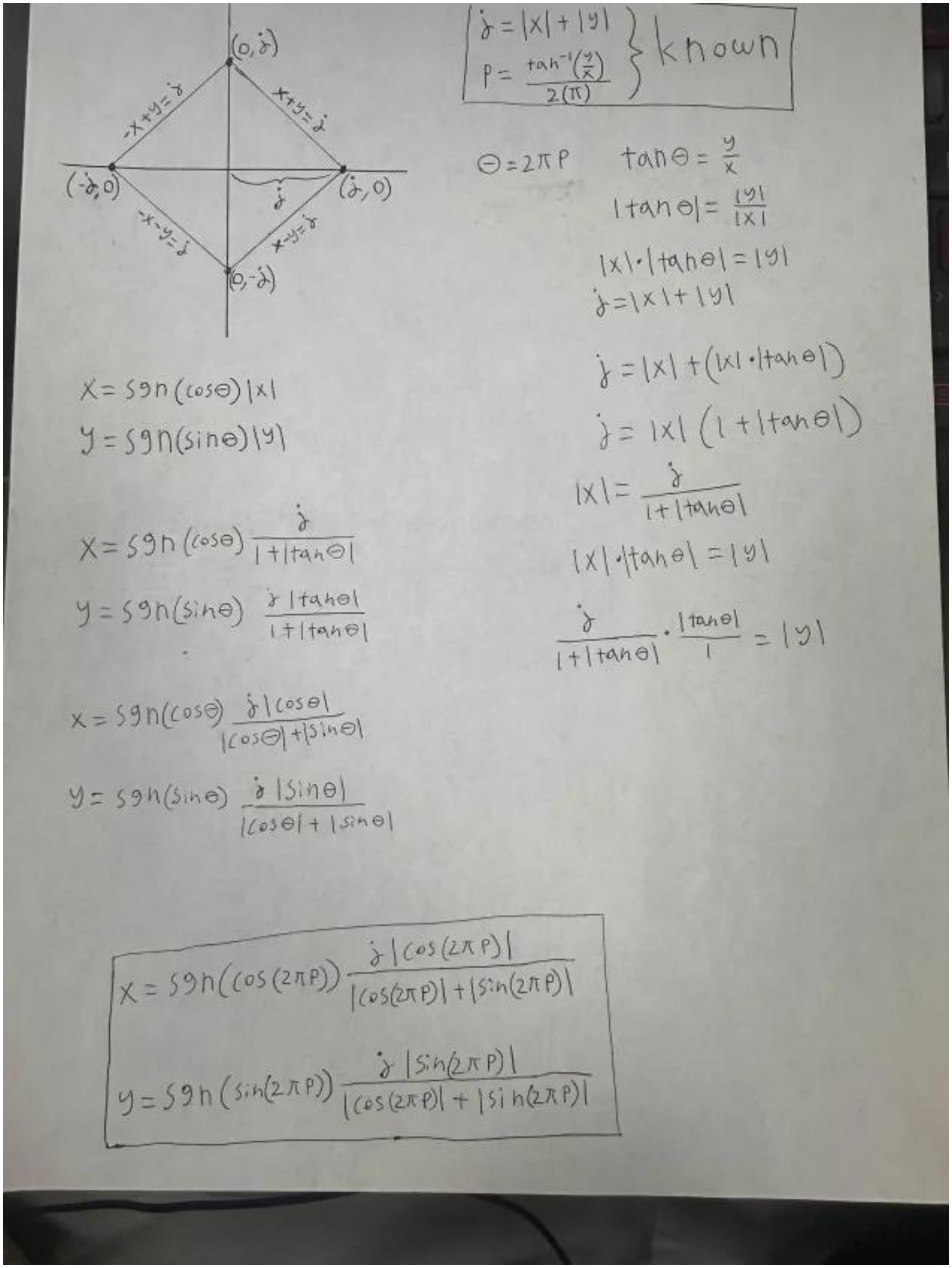
Appendix 1: Algebraic Derivation (Isolation of *x* and *y*) (Author 1 & Author 2, 2024)



Appendix 2: Derivation (Partial Derivative $\frac{\partial x}{\partial p}$) (Author 1 & Author 2, 2024)